Problems from Lecture notes on Classical Mechanics and Electromagnetism in Accelerator Physics

From lecture notes of the
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Lecture 2

Linear and Nonlinear Oscillators

Problem 2.1. *Prove that*

$$x(t) = x_0 \cos\omega_0 t + \frac{\dot{x}_0}{\omega_0} \sin\omega_0 t + \frac{1}{\omega_0}\int_0^t \sin\omega_0(t-t')f(t')dt' \ , \tag{2.1}$$

gives a solution to

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = f(t) \ , \tag{2.2}$$

with $\gamma = 0$. *Verify that initial conditions are satisfied. Generalize the solution Eq. (2.1) for the case when* $\gamma \neq 0$.

Problem 2.2. *The function* $f(t)$ *is shown in Fig. 2.1 (next page): it is equal to zero for* $t < -\Delta t$, *and is constant for* $t > \Delta t$ *with a smooth transition in between. Describe the behavior of the linear oscillator driven by this force in the limits* $\Delta t \ll \omega_0^{-1}$ *and* $\Delta t \gg \omega_0^{-1}$.

Problem 2.3. *Prove that*

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0 \tag{2.3}$$

conserves the quantity $x^2(t) + \dot{x}(t)^2/\omega_0^2$.

Problem 2.4. *Assume* $\gamma = 0$ *in the equation*

$$\frac{d^2\xi}{dt^2} + \gamma\frac{d\xi}{dt} + \omega_0^2\xi = f_0 e^{-i\omega t} \, . \tag{2.4}$$

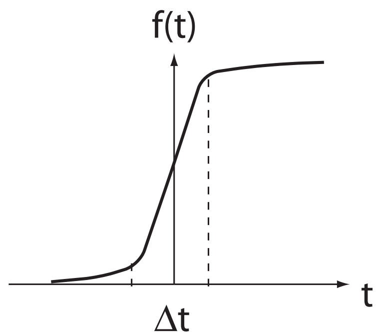

Figure 2.1: Function $f(t)$.

Show that if $\omega \gg \omega_0$ then one can neglect the term $\omega_0^2\xi$ in the equation. In other words, the oscillator responds to the driving force as a free particle. This fact explains why the dielectric response of many media to x-rays can be computed neglecting the binding of electrons to nuclei.

Problem 2.5. *Draw a phase portrait of a linear oscillator with and without damping.*

Problem 2.6. *Derive the following equation:*

$$\omega \approx \omega_0 \left(1 - \frac{\theta_0^2}{16}\right) , \tag{2.5}$$

directly from

$$\frac{1}{2} T\omega_0 = \frac{1}{\sqrt{2}} \int_{-\theta_0}^{\theta_0} \frac{d\theta}{\sqrt{(\cos\theta - \cos\theta_0)}} . \tag{2.6}$$

Problem 2.7. *Fig. 2.2 (next page) shows a numerically computed trajectory for a pendulum with $\omega_0 = 1$. Try to figure out what is the energy E for this trajectory and explain qualitatively the shape of the curve.*

Problem 2.8. *Verify that*

$$a = -\frac{3\beta}{8\omega_0} - \frac{5\alpha^2}{12\omega_0^3} , \tag{2.7}$$

gives the result

$$\omega \approx \omega_0 \left(1 - \frac{\theta_0^2}{16}\right) , \tag{2.8}$$

for the pendulum.

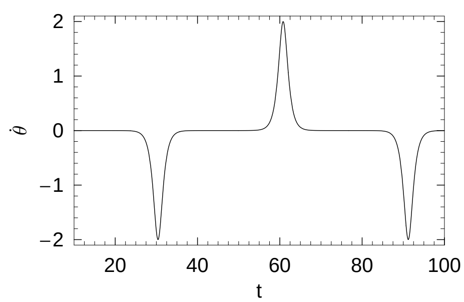

Figure 2.2: Dependence of $\dot{\theta}$ versus time for a pendulum trajectory.

Lecture 3

Lagrangian and Hamiltonian equations of motion

Problem 3.1. *For a linear oscillator, the Lagrangian is*

$$L = \frac{m}{2}\dot{x}^2 - \frac{m\omega^2}{2}x^2 \, .$$

Find equations of motion.

Problem 3.2. *Consider a pendulum of length l and mass m, supported by a pivot that is driven in the vertical direction by a given function of time $y_s(t)$. Obtain the Lagrangian and derive equations of motion for the pendulum (Ref. G. Sussman and J. Wisdom. Structure and Interpretation of Classical Mechanics. MIT Press, 2001, page 49).*

Problem 3.3. *For the system in Problem 3.2, analyze particle's motion in a rotating frame using the Lagrangian approach (Ref. J. Josè and E. Saletan. Classical dynamics: a contemporary approach. Cambridge University Press, 1998, pages 74-76).*

Problem 3.4. *Derive the following equations of motion:*

$$\frac{d\boldsymbol{p}}{dt} = q\boldsymbol{E} + q\boldsymbol{v} \times \boldsymbol{B} \, , \tag{3.1}$$

from the Lagrangian

$$L(\boldsymbol{r}, \boldsymbol{v}, t) = -mc^2\sqrt{1 - v^2/c^2} + e\boldsymbol{v} \cdot \boldsymbol{A}(\boldsymbol{r}, t) - e\phi(\boldsymbol{r}, t) \, . \tag{3.2}$$

Problem 3.5. *Write the same Lagrangian,*

$$L(\boldsymbol{r}, \boldsymbol{v}, t) = -mc^2\sqrt{1 - v^2/c^2} + e\boldsymbol{v} \cdot \boldsymbol{A}(\boldsymbol{r}, t) - e\phi(\boldsymbol{r}, t) \ . \tag{3.3}$$

in the cylindrical coordinate system with z directed along the magnetic field. Derive the equations of motion.

Problem 3.6. *Do the same for the coordinate system (x', s, z) shown in Fig. 3.1.*

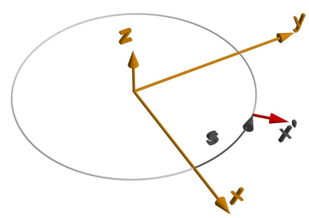

Figure 3.1: The coordinate system x', s, z. The circle radius is equal to the Larmor radius R. The coordinate x' is defined as a difference between the polar radius r and the circle radius R.

Problem 3.7. *The magnetic field given by*

$$\boldsymbol{B} = (0, 0, B_0) \ , \tag{3.4}$$

can be represented by a different vector potential $\boldsymbol{A} = \frac{1}{2}(-B_0 y, B_0 x, 0)$. Show that the equations of motion are the same as for the vector potential used in the notes,

$$\boldsymbol{A} = (-B_0 y, 0, 0) \, . \tag{3.5}$$

Problem 3.8. *Find conjugate momenta in cylindrical coordinates of a charged particle moving in an electromagnetic field, using the Hamiltonian*

$$H = \sqrt{(mc^2)^2 + c^2(\boldsymbol{\pi} - e\boldsymbol{A})^2} + e\phi \ . \tag{3.6}$$

Problem 3.9. *The angular momentum $\boldsymbol{M}$ of a particle is defined as $\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{p}$. Find the Poisson brackets $\{M_i, x_k\}$, $\{M_i, p_k\}$ and $\{M_i, M_k\}$, where the indices i and k take the values x, y and z.*

Problem 3.10. *Simplify L and H (for a particle in an electromagnetic field) in the nonrelativistic limit $v \ll c$.*

Lecture 4

Canonical transformations

Problem 4.1. *Show that the transformation* $P_i = \lambda p_i$, $Q_i = q_i$, $H' = \lambda H$, *where* λ *is a constant parameter, preserves the Hamiltonian structure of equations.*

Problem 4.2. *Using the Poisson brackets prove that the transformations*

$$Q_i = p_i \,, \qquad P_i = -q_i \ , \tag{4.1}$$

and

$$Q_i = p_i \,, \qquad P_i = -q_i \ , \tag{4.2}$$

are canonical.

Problem 4.3. *Find generating functions for the transformations*

$$Q_i = p_i \,, \qquad P_i = -q_i \tag{4.3}$$

and

$$Q_i = p_i \,, \qquad P_i = -q_i \,. \tag{4.4}$$

Problem 4.4. *Find generating functions for the* contact *transformation*

$$Q_i = f_i(q_1, q_2, \ldots, q_n) \,. \tag{4.5}$$

Problem 4.5. *Find the generating function of the third type for the transformation*

$$Q_i = q_i \,, \qquad P_i = p_i \,. \tag{4.6}$$

This problem illustrates the fact that the choice of the type of the generating function is not unique.

Problem 4.6. *From equations*

$$J = \frac{1}{2\omega}\left(\omega^2 x^2 + p^2\right) . \tag{4.7}$$

and

$$\phi = -\arctan\frac{p}{\omega x} . \tag{4.8}$$

express x *and* p *through* J *and* ϕ*. Verify that the result agrees with equations*

$$x = a(J)\cos\phi\,, \qquad p = -a(J)\omega\sin\phi\,. \tag{4.9}$$

Lecture 5

Liouville's theorem. Action-angle variables.

Problem 5.1. *Derive*

$$M J_{2n} M^T = J_{2n} \tag{5.1}$$

for $n = 2$.

Problem 5.2. *Find the action-angle variables for the system with the following potential*

$$U(x) = \begin{cases} \infty, & x < 0 \\ Fx, & x > 0 \end{cases} \tag{5.2}$$

Problem 5.3. *Prove that $|\det M| = 1$, where*

$$|\det\ M| = \left| \frac{dQ_1 dQ_2 \ldots dQ_n dP_1 dP_2 \ldots dP_n}{dq_1 dq_2 \ldots dq_n dp_1 dp_2 \ldots dp_n} \right| \tag{5.3}$$

as the result of Hamiltonian flow.

Lecture 6

Coordinate system and Hamiltonian in an accelerator

Problem 6.1. *Check that equations*

$$
\begin{aligned}
\frac{d\boldsymbol{r}_0}{ds} &= \hat{\boldsymbol{s}}\,, \\
\frac{d\hat{\boldsymbol{s}}}{ds} &= -\frac{\hat{\boldsymbol{x}}}{\rho(s)}\,, \\
\frac{d\hat{\boldsymbol{x}}}{ds} &= \frac{\hat{\boldsymbol{s}}}{\rho(s)}\,, \\
\frac{d\hat{\boldsymbol{y}}}{ds} &= 0\,.
\end{aligned} \tag{6.1}
$$

hold for a circular orbit.

Problem 6.2. *Fig. 6.1 shows the electron trajectory in a four-dipole chicane (typically used for bunch compressions). Indicate the direction of axis x assuming that the y axis is directed out of the page. Determine the sign of the orbit radius ρ and the magnetic field direction of each of four dipoles along the orbits. What happens with this sign if the particle is moving in the direction opposite to the one shown in the figure?*

Problem 6.3. *Verify that from the definition of ρ in Eqs. (6.1) it follows that the sign in equation*

$$
\rho(s) = \frac{p}{eB_y(s)}\,. \tag{6.2}
$$

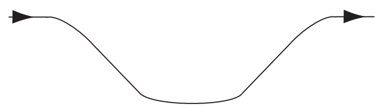

Figure 6.1: Electron trajectory in a chicane. Assume that the y axis is directed out of the page.

is correct for arbitrary sign of the charge e and the direction of motion in the reference orbit.

Problem 6.4. *Verify that Eqs. (6.1) and equations*

$$\nabla\phi = \hat{\boldsymbol{x}}\frac{\partial\phi}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial\phi}{\partial y} + \hat{\boldsymbol{s}}\frac{1}{1+x/\rho}\frac{\partial\phi}{\partial s}\,, \tag{6.3}$$

$$(\nabla\times\boldsymbol{A})_x = -\frac{1}{1+x/\rho}\frac{\partial A_y}{\partial s} + \frac{\partial A_s}{\partial y}\,, \tag{6.4}$$

$$(\nabla\times\boldsymbol{A})_s = -\frac{\partial A_x}{\partial y} + \frac{\partial A_y}{\partial x}\,, \tag{6.5}$$

$$(\nabla\times\boldsymbol{A})_y = -\frac{1}{1+x/\rho}\frac{\partial A_s(1+x/\rho)}{\partial x} + \frac{1}{1+x/\rho}\frac{\partial A_x}{\partial s}\,, \tag{6.6}$$

$$\nabla\cdot\boldsymbol{A} = \frac{1}{1+x/\rho}\frac{\partial A_x(1+x/\rho)}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{1}{1+x/\rho}\frac{\partial A_s}{\partial s}\,. \tag{6.7}$$

hold for a circular orbit.

Problem 6.5. *Find the Hamiltonian K for the following model Hamiltonian H:*

$$H(x,\Pi_x,s,\Pi_s) = \frac{\Pi_x^2}{2} + \omega^2(s)\frac{x^2}{2} + v\Pi_s\,, \tag{6.8}$$

where v is a constant. Prove that both Hamiltonians describe the same dynamics.

Problem 6.6. *Make canonical transformation $(t,-h)\to(z_t,p)$ using the generating function $F_2(p,t) = -ct\sqrt{p^2+m^2c^2}$. Explain the meaning of the new variables.*

Lecture 7

Equations of motion in an accelerator

Problem 7.1. *The magnetic field $B_s(s)$ of a solenoid cannot be described with a single longitudinal component A_s of the vector potential. Show that this magnetic field can be represented with the vector potential that has two transverse components:*

$$A_x = -B_s y/2\,, \qquad A_y = B_s x/2\,. \tag{7.1}$$

.

Problem 7.2. *Using*

$$\frac{dt}{ds} = \frac{\partial \Pi_s}{\partial H} = -\frac{\partial K}{\partial h} \tag{7.2}$$

and the Hamiltonian $\mathcal{H}$ given in the lecture notes on p.60 as Equation (7.11), show that

$$\frac{dt}{ds} = \frac{1}{v}\left(1 + \frac{x}{\rho}\right)\,. \tag{7.3}$$

Problem 7.3. *Find terms in the Hamiltonian $\mathcal{H}$ responsible for the skew quadrupole (the magnetic field given by the equation $\boldsymbol{B} = G_s(s)(-\hat{\boldsymbol{y}}y + \hat{\boldsymbol{x}}x)$).*

Problem 7.4. *Using the vector potential*

$$A_x = -B_s y/2\,, \qquad A_y = B_s x/2 \tag{7.4}$$

for the solenoid and starting from the Hamiltonian

$$K = -\left(1+\frac{x}{\rho}\right)\left[\frac{1}{c^2}h^2 - (P_x - eA_x)^2 - (P_y - eA_y)^2 - m^2c^2\right]^{1/2} - eA_s\left(1+\frac{x}{\rho}\right). \tag{7.5}$$

find the contribution to $\mathcal{H}$ of the magnetic field of the solenoid. [Hint: assume that B_s is small and use the Taylor expansion in the Hamiltonian (7.5) keeping linear terms and second order terms in B_s.]

Problem 7.5. *Derive equation*

$$\frac{1}{2}\beta\beta'' - \frac{1}{4}\beta'^2 + K\beta^2 = 1 \tag{7.6}$$

from equation

$$w'' - \frac{1}{w^3} + K(s)w = 0\,. \tag{7.7}$$

Problem 7.6. *Find solution of equation*

$$\frac{1}{2}\beta\beta'' - \frac{1}{4}\beta'^2 + K\beta^2 = 1 \tag{7.8}$$

in free space where $K = 0$.

Problem 7.7. *Calculate a jump of the derivative of the beta function through a thin quadrupole. Such a quadrupole is defined by $K(s) = f^{-1}\delta(s - s_0)$, where f is called the focal length of a thin quadrupole.*

Problem 7.8. *A FODO lattice is a sequence of thin quadrupoles with alternating polarities:*

$$K_{\mathrm{FODO}}(s) = \sum_{n=-\infty}^{\infty} K_0\delta(s - nl) - K_0\delta\left(s - \left[n + \frac{1}{2}\right]l\right), \tag{7.9}$$

where l is the period of the lattice. Solve equation

$$\frac{1}{2}\beta\beta'' - \frac{1}{4}\beta'^2 + K\beta^2 = 1 \tag{7.10}$$

for the FODO lattice and find $\beta(s)$. For a given value of l find the maximum value of K_0 for which the motion is stable.

Problem 7.9. *Consider two rings with circumferences C_1 and C_2. Assume that $C_1 = \lambda C_2$ and $K_2(s) = \lambda^2 K_1(\lambda s)$, and prove that $\beta_2 = \lambda^{-1}\beta_1(\lambda s)$.*

Lecture 8

Action-angle variables for circular machines

Problem 8.1. *Using equations*

$$
\begin{aligned}
-\tan\phi &= \frac{\beta P_x}{x} + \alpha \\
\alpha &= -\frac{\beta'}{2} \\
J = \frac{1}{2\beta}&\left[x^2 + (\beta P_x + \alpha x)^2\right] \ ,
\end{aligned}
\tag{8.1}
$$

show by direct calculation of Poisson brackets that the transformation $(x, P_x) \to (\phi, J)$ is canonical.

Problem 8.2. *Find the major and minor half axes, and the tilt of the ellipse shown below (next page).*

Problem 8.3. *Prove that the transformation $(x, P_x) \to (\bar{x}, \bar{P}_x)$ with*

$$
\bar{x} = \frac{1}{\sqrt{\beta}} x \ , \qquad \bar{P}_x = \frac{1}{\sqrt{\beta}}(\beta P_x + \alpha x)
\tag{8.2}
$$

is canonical. Prove that phase space orbits plotted in variables $\bar{x}$, $\bar{P}_x$ are circles.

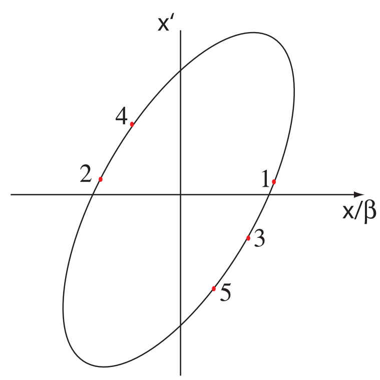

Figure 8.1: Phase space ellipse and a particle's positions at consecutive turns.

Lecture 9

Field errors and nonlinear resonances

Problem 9.1. *Starting from the Hamiltonian*

$$\mathcal{H} = \frac{1}{2}x'^2 + \frac{1}{2}K(s)x^2 + \frac{e\Delta B(s)}{p}x \tag{9.1}$$

transform to the action-angle variables using the following generating function

$$F_1(x, \phi, s) = \frac{[x - x_0(s)]^2}{2\beta}\left(\frac{\beta'}{2} - \tan\phi\right) + x x_0'(s)\ . \tag{9.2}$$

Show that in this case

$$J(x, P_x, s) = \frac{1}{2\beta}\left\{(x - x_0)^2 + \left[\beta\left(P_x - x_0'\right) + \alpha(x - x_0)\right]^2\right\}, \tag{9.3}$$

and obtain the Hamiltonian

$$\hat{\mathcal{H}} = \frac{J}{\beta}\,. \tag{9.4}$$

Problem 9.2. *What is the effect on the beam orbit of the error magnetic field $\Delta B_x(s)$ in the horizontal plane?*

Problem 9.3. *Solve equation*

$$V(\phi, I, s) + \frac{1}{\beta}G_\phi + G_s = 0\,. \tag{9.5}$$

for the function G. Hint: seek solution in the form $G(\phi, I, s) = \mathrm{Re}(\hat{G}(I, s)e^{2i\phi})$.

Problem 9.4. *Verify by direct calculation that G given by*

$$G = -\frac{I}{4\sin 2\pi\nu}\int_s^{s+C} ds'\Delta K(s')\beta(s')\sin 2(\phi - \psi(s) + \psi(s') - \pi\nu) \tag{9.6}$$

satisfies equation

$$V(\phi, I, s) + \frac{1}{\beta}G_\phi + G_s = 0\,. \tag{9.7}$$

Problem 9.5. *Show that the tune change is given by the following equation*

$$\Delta\nu = \frac{1}{4\pi}\int_0^C ds\Delta K(s)\beta(s)\ . \tag{9.8}$$

Recall that

$$\nu = \frac{1}{2\pi}\int_0^C \frac{ds}{\beta(s)}. \tag{9.9}$$

Problem 9.6. *Calculate the beta beat and the tune change for a localized perturbation of the lattice: $\Delta k = \Delta K_0 \delta(s - s_0)$.*

Lecture 10

Resonance overlapping and dynamic aperture

Problem 10.1. *Prove that the standard map defines a canonical transformation* $(I_n, \theta_n) \to (I_{n+1}, \theta_{n+1})$.

Problem 10.2. *Prove the following property of the standard map: for two trajectories starting from the same initial value* θ_0 *but with different values* $I_0^{(1)}$ *and* $I_0^{(2)}$*, such that* $I_0^{(2)} - I_0^{(1)} = 2\pi m$*, where* m *is an integer, the difference* $I_n^{(2)} - I_n^{(1)}$ *remains equal to* $2\pi m$ *for all values of* n.

Problem 10.3. *Prove that equations*

$$J = I - 2\pi n\,, \qquad \phi = \theta - 2\pi n t\,. \tag{10.1}$$

define a canonical transformation, find the corresponding generating function F_2 *and obtain the Hamiltonian*

$$\mathcal{H}' = \frac{1}{2} J^2 + K \cos\phi + \text{const}\,. \tag{10.2}$$

Lecture 11

The kinetic equation

Problem 11.1. *Write the Vlasov equation for a beam distribution* $f(x, P_x, s)$ *in terms of variables* x *and* P_x.

Problem 11.2. *Give a direct proof that the function*

$$f = const\ e^{-J/\epsilon_0} = const \exp\left(-\frac{1}{2\beta\epsilon_0}\left[x^2 + (\beta P_x + \alpha x)^2\right]\right) \tag{11.1}$$

satisfies the Vlasov equation.

Problem 11.3. *Action and angle variables are more convenient for a study of phase mixing. Use these variables and find the limit of the distribution function integrated over* δ *in the limit* $t \to \infty$.

Lecture 13

Primer in special relativity

Problem 13.1. *Derive the Lorentz transformation when velocity $\boldsymbol{v}$ is at 45° to the z axis, $\boldsymbol{v} = v(0, 1/\sqrt{2}, 1/\sqrt{2})$.*

Problem 13.2. *Using the matrix formalism, show that the inverse Lorentz transformation is given by the following equations:*

$$
\begin{aligned}
x' &= x \ , \\
y' &= y \ , \\
z' &= \gamma(z - \beta c t) \ , \\
t' &= \gamma(t - \beta z/c) \ .
\end{aligned}
\tag{13.1}
$$

Explain the meaning of the minus sign in front of β.

Problem 13.3. *A muon at rest has a mean life time of 2.2 μs. To what energy one needs to accelerate the muon in order to get the life time (in the lab frame) of 1 ms. The muon mass corresponds to 105 MeV.*

Problem 13.4. *A bunch of 10^{10} electrons with energy 15 GeV has a length of 100 micron and a radius of 30 micron (in the lab frame). What is the electron density (in units of particles per cubic centimeter) in the beam frame?*

Problem 13.5. *Using the equations*

$$
\begin{aligned}
k_x &= {k_x}' \ , \\
k_y &= {k_y}' \ , \\
k_z &= \gamma({k_z}' + \beta\omega'/c) \ , \\
\omega &= \gamma(\omega' + \beta c {k_z}') \ ,
\end{aligned}
\tag{13.2}
$$

prove that $\omega = ck$ follows from $\omega' = ck'$.

Problem 13.6. *Prove that*

$$\cos\theta' = \frac{\cos\theta - \beta}{1 - \beta\cos\theta}\,, \qquad \sin\theta' = \frac{\sin\theta}{\gamma(1 - \beta\cos\theta)}\,. \tag{13.3}$$

Problem 13.7. *A laser light of frequency ω copropagates with a relativistic beam with $\gamma \gg 1$. Find the laser frequency in the beam frame.*

Problem 13.8. *Consider Lorentz transformation of fields in a plane electromagnetic wave propagating along axis z. The electric field is directed along x and the magnetic field is directed along y with $E_x = cB_y$. Derive the transformation formula for the absolute value of the Poynting vector of the wave.*

Problem 13.9. *An electromagnetic wave with the frequency ω and the electric field amplitude E_0 occupies a volume with dimensions $L_x \times L_y \times L_z$. It propagates along the z axis with fields that satisfy $E_x = cB_y$. Using results of the previous problem find the electromagnetic energy W of the wave in the lab frame and the energy W' in a frame K' moving with velocity v relative to the lab frame. Show that $W/\omega = W'/\omega'$, where ω' is the frequency of the wave in K'.*

Lecture 14

Selected electrostatic and magnetostatic problems

Problem 14.1. *Show that at large distances from the center, the equation*

$$\phi = \frac{1}{4\pi\epsilon_0}\sqrt{\frac{2}{\pi}}\frac{Q}{\sigma_x\sigma_y\sigma_z}\int_0^\infty d\lambda \frac{e^{-\frac{x^2\lambda^2}{2(\lambda^2\sigma_x^2+1)}}e^{-\frac{y^2\lambda^2}{2(\lambda^2\sigma_y^2+1)}}e^{-\frac{z^2\lambda^2}{2(\lambda^2\sigma_z^2+1)}}}{\sqrt{\lambda^2+\sigma_x^{-2}}\sqrt{\lambda^2+\sigma_y^{-2}}\sqrt{\lambda^2+\sigma_z^{-2}}}\,. \tag{14.1}$$

reduces to

$$\phi = \frac{Q}{4\pi\epsilon_0\sqrt{x^2+y^2+z^2}}\,. \tag{14.2}$$

Problem 14.2. *Prove by direct calculation that the potential given by*

$$\phi(x,y,z) = \frac{Q}{2}\int_{\lambda(x,y,z)}^\infty \frac{d\lambda'}{\sqrt{(a^2+\lambda')(b^2+\lambda')(c^2+\lambda')}} \tag{14.3}$$

satisfies the Laplace equation.

Lecture 15

Self field of a relativistic beam

Problem 15.1. *Verify by direct calculation that equations*

$$\begin{aligned} \boldsymbol{E} &= -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t} \\ \boldsymbol{B} &= \nabla \times \boldsymbol{A} \end{aligned} \tag{15.1}$$

applied to the potentials

$$\phi = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathcal{R}}\,, \qquad \boldsymbol{A} = \frac{Z_0}{4\pi}\boldsymbol{\beta}\frac{q}{\mathcal{R}}\,. \tag{15.2}$$

give the fields

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\frac{q\boldsymbol{r}}{\gamma^2\mathcal{R}^3}\,. \tag{15.3}$$

and

$$\boldsymbol{B} = \frac{1}{c^2}\boldsymbol{v}\times\boldsymbol{E}\,. \tag{15.4}$$

Problem 15.2. *Given the form of the electric field*

$$\begin{aligned} \boldsymbol{E} &= -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t} \\ \boldsymbol{B} &= \nabla \times \boldsymbol{A}\,, \end{aligned} \tag{15.5}$$

make a plot of the dependence of E versus θ, where θ is the angle between $\boldsymbol{r}$ and the (x, y) plane. Assume $\gamma \gg 1$.

Problem 15.3. *Using equations*

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\frac{q\boldsymbol{r}}{\gamma^2\mathcal{R}^3}\,. \tag{15.6}$$

show that in the limit $\gamma \to \infty$ the following relations hold

$$\int_{-\infty}^{\infty} E_x dz = \frac{1}{4\pi\epsilon_0}\frac{2qx}{\rho^2}\,, \qquad \int_{-\infty}^{\infty} E_y dz = \frac{1}{4\pi\epsilon_0}\frac{2qy}{\rho^2}\,. \tag{15.7}$$

Problem 15.4. *Derive the equation*

$$\mathcal{E}_\parallel(z,z') = -\frac{1}{4\pi\epsilon_0}\frac{2}{a^2}(z-z')\left(\frac{1}{\sqrt{a^2/\gamma^2+(z-z')^2}} - \frac{1}{|z-z'|}\right)\,. \tag{15.8}$$

for $\mathcal{E}_\parallel$ and analyze it considering limits $|z-z'| \ll a/\gamma$ and $|z-z'| \gg a/\gamma$. Hint: represent a thin charged disk as a collection of infinitesimally small rings.

Problem 15.5. *Derive an expression for the field $E_\parallel(z)$ on the beam axis for a Gaussian bunch using the result of Section 14.1 in Lecture 14. Assume $\sigma_x = \sigma_y$.*

Problem 15.6. *A bunch of particles in a future linear collider will have a charge of about 1 nC, bunch length $\sigma_z \approx 200$ μm, and will be accelerated in the linac from 5 GeV to 250 GeV. Estimate the energy spread in the beam induced by the the space charge, assuming the bunch radius of 50 μm.*

Lecture 16

Effect of environment on electromagnetic field of a beam

Problem 16.1. *Calculate the skin depth in copper ($\sigma = 5.8 \cdot 10^7$ 1/Ohm·m) and stainless steel ($\sigma = 1.4 \cdot 10^6$ 1/Ohm·m) at the frequency of 5 GHz.*

Problem 16.2. *Given the tangential component $B_0 e^{-i\omega t}$ of the magnetic field on the surface, find the averaged over time energy absorbed in the metal per unit time per unit area. Hint: compute the averaged over time z-component of the Poynting vector on the surface. Answer: $\omega\delta B_0^2/4\mu_0$.*

Problem 16.3. *Find how the Leontovich boundary conditions transform into a frame moving with relativistic velocity v parallel to the metal surface in the direction perpendicular to the magnetic field (beam frame).*

Lecture 17

Plane electromagnetic waves and Gaussian beams

Problem 17.1. *At time $t = 0$ the electromagnetic field in free space is given by functions $\boldsymbol{E}_0(\boldsymbol{r})$ and $\boldsymbol{B}_0(\boldsymbol{r})$ (note that $\nabla \cdot \boldsymbol{E}_0 = \nabla \cdot \boldsymbol{B}_0 = 0$). Find the field at time t. [Hint: represent $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{B}(\boldsymbol{r}, t)$ as integrals over plane waves.]*

Problem 17.2. *A plane electromagnetic wave propagates at some angle in a frame moving with velocity βc along the z axis. The magnitude of the Poynting vector at some location in the wave is equal to S'. Show that in the laboratory frame the magnitude of the Poynting at this location is given by the following equation*

$$S = \frac{S'}{\gamma^2(1 - \beta \cos\theta)^2}\,, \tag{17.1}$$

where θ is the angle between the direction of propagation in the lab frame and the z axis.

Problem 17.3. *Prove that the function $u(r, t) = f(r - ct)/r$ where r is the distance to the origin of the coordinate system, satisfies the scalar wave equation $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2 - (1/c^2)\partial^2 u/\partial t^2 = 0$, if $r > 0$.*

Problem 17.4. *Calculate the longitudinal electric field E_z for a Gaussian laser beam using the equation $\nabla \cdot \boldsymbol{E} = 0$.*

Problem 17.5. *Show that the energy flux (the Poynting vector integrated over the cross section of the beam) of a Gaussian laser beam is equal to*

$$\frac{\pi}{4Z_0} E_0^2 w_0^2\,. \tag{17.2}$$

Problem 17.6. *A laser pulse has an energy of 1 J and duration 100 fs. It is focused into a spot of radius 10 μm. Find the magnitude of the electric field in the focus.*

Problem 17.7. *Expand the laser field over plane waves, for the laser pulse described in the previous problem.*

Lecture 18

Radiation and retarded potentials

Problem 18.1. *Find solutions of equation*

$$c^2 t'^2 = (z - v(t-\tau))^2 + x^2 + y^2 \,, \tag{18.1}$$

and analyze them.

Problem 18.2. *Find $\partial R/\partial t$ and ∇R. Show that*

$$\nabla t_{\mathrm{ret}} = -\frac{\boldsymbol{n}}{c(1 - \boldsymbol{n} \cdot \boldsymbol{\beta}_{\mathrm{ret}})} \,. \tag{18.2}$$

The operator ∇ here is understood as $\hat{\boldsymbol{x}}\partial/\partial x + \hat{\boldsymbol{y}}\partial/\partial y + \hat{\boldsymbol{z}}\partial/\partial z$.

Problem 18.3. *A point charge is at rest for $t < 0$. It is then uniformly accelerated during time interval Δt with acceleration a, and moves with a constant velocity $v = a\Delta t$ at $t > \Delta t$. Using the retarded potentials find the electromagnetic field in space at $t > \Delta t$. Assume $v \ll c$.*

Problem 18.4. *Verify that the Liénard-Wiechert potentials can now be derived from the retarded potentials assuming $\rho(\boldsymbol{r}, t) = q\delta(\boldsymbol{r} - \boldsymbol{r}_0(t))$ and $\boldsymbol{j}(\boldsymbol{r}, t) = q\boldsymbol{v}(t)\delta(\boldsymbol{r} - \boldsymbol{r}_0(t))$ with $\boldsymbol{v} = d\boldsymbol{r}_0/dt$.*

Lecture 19

Scattering of electromagnetic waves

Problem 19.1. *Prove that the ratio $qE_0/m\omega c$ is a Lorentz invariant—it does not change under the Lorentz transformation (in other words, it is the same in any coordinate system moving relative to the laboratory reference frame).*

Problem 19.2. *Prove the equation*

$$|\boldsymbol{n} \times \boldsymbol{E_0}|^2 = E_0^2(1 - \sin^2\theta\cos^2\phi)\,. \tag{19.1}$$

Problem 19.3. *Consider scattering of an electromagnetic wave on a charge q that is attached to an immobile point through a spring, and can oscillate with the frequency ω_0. Find the scattering cross section as a function of frequency of the incident wave ω.*

Problem 19.4. *In the derivation above (about light pressure, p.155 in the notes), we neglected the term $q\boldsymbol{v} \times \boldsymbol{B}$ where $\boldsymbol{v}$ is given by the real part of equation*

$$\boldsymbol{v} = \frac{iq}{m\omega}\boldsymbol{E_0}e^{-i\omega t}\ . \tag{19.2}$$

Show that $\langle \boldsymbol{v} \times \boldsymbol{B} \rangle = 0$.

Problem 19.5. *Estimate the pressure of the solar light on the surface of the Earth. The solar radiation power is about 1 kW/m^2.*

Problem 19.6. *R. Ruth and Z. Huang proposed to use Thomson scattering in a compact electron ring as a source of intense X-ray radiation (PRL* **80** *976*

(1998)). The electron energy in the ring is 8 MeV, the number of electron in the bunch is $1.1 \cdot 10^{10}$*, the laser energy is* 20 *mJ, the laser pulse length is* 1 *mm, and the laser is focused to the spot size* 25 *micron. Estimate the number of photons from a single collision of the laser pulse with the electron beam.*

Lecture 20

Synchrotron radiation

Problem 20.1. *Find asymptotic dependence $B_y(t)$ for $|t - r/c| \gg \rho/c\gamma^3$, given*

$$B_y = \frac{Z_0 q}{\pi r \rho} \frac{\gamma^{-2} - \xi^2}{(\xi^2 - \gamma^{-2})^3} \ . \tag{20.1}$$

Problem 20.2. *Prove that the area under the curve $B_y(t)$ is equal to zero (that is $\int_{t=-\infty}^{t=\infty} B_y(t)dt = 0$), again with*

$$B_y = \frac{Z_0 q}{\pi r \rho} \frac{\gamma^{-2} - \xi^2}{(\xi^2 - \gamma^{-2})^3} \ . \tag{20.2}$$

Problem 20.3. *Simplify the equation*

$$\frac{d^2\mathcal{W}}{d\omega d\Omega} = \frac{q^2 Z_0}{12\pi^3} \left(\frac{\rho\omega}{c}\right)^2 \left(\frac{1}{\gamma^2} + \psi^2\right)^2 \left[K_{2/3}^2(\zeta) + \frac{\psi^2}{1/\gamma^2 + \psi^2} K_{1/3}^2(\zeta)\right] , \tag{20.3}$$

in the limit $\psi \gg 1/\gamma$. Here,

$$\zeta \equiv \frac{\omega\rho}{3c} \left(\frac{1}{\gamma^2} + \psi^2\right)^{3/2} \ . \tag{20.4}$$

Make a plot of the quantity $\omega^{-2/3} d^2\mathcal{W}/(d\omega d\Omega)$ versus the quantity $\omega\rho\psi^3/c$. Infer from these equations that the angular spread of the radiation at frequency $\omega \ll \omega_c$ is of order of $(c/\omega\rho)^{1/3}$.

Problem 20.4. *Calculate RF power needed to compensate the synchrotron radiation in the High Energy Ring of PEP-II. The parameters for the PEP-II high energy ring are: beam energy $E = 9$ GeV, average current $I = 1$ A, and $\rho = 174$ m in the bends. [Hint: because average current and not total charge is specified, the actual circumference is irrelevant; only the total length of the bends matters.]*

Lecture 21

Undulator radiation

Problem 21.1. *Integrate the equation*

$$\frac{d\mathcal{P}}{d\Omega} = \frac{Z_0}{4\pi^2}\frac{q^4\gamma^4 B_0^2}{m^2}\frac{(1+\gamma^2\theta^2)^2 - 4\theta^2\gamma^2\cos^2\phi}{(1+\gamma^2\theta^2)^5} \tag{21.1}$$

over ϕ and find $d\mathcal{I}/d\theta$. Using the relation

$$\omega = \frac{2\gamma^2 k_u c}{1+\gamma^2\theta^2} \tag{21.2}$$

between the frequency and the angle show that the intensity of the radiation per unit frequency is

$$\frac{d\mathcal{P}}{d\omega} = \frac{3\mathcal{P}_0}{\omega_0}\frac{\omega}{\omega_0}\left(2\left(\frac{\omega}{\omega_0}\right)^2 - 2\left(\frac{\omega}{\omega_0}\right) + 1\right) \tag{21.3}$$

for $\omega < \omega_0$ and zero for $\omega > \omega_0$. The plot of this function is shown in the figure below.

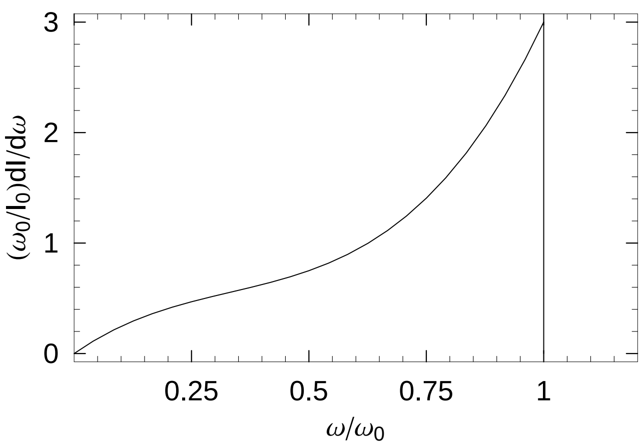

Figure 21.1: The spectrum of the undulator radiation given by Eq. (21.3).

Lecture 22

Transition and diffraction radiation

Problem 22.1. *Draw a picture of field lines at time $t > 0$ for transition radiation.*

Problem 22.2. *The usual setup in the experiment for the optical transition radiation (OTR) diagnostic is shown in Fig. 22.1: the beam passes through a metal foil tilted at an angle of 45 degrees relative to the beam orbit. Show that in this case the radiation propagates predominantly in the direction perpendicular to the orbit. How to solve this problem using the method of image charges?*

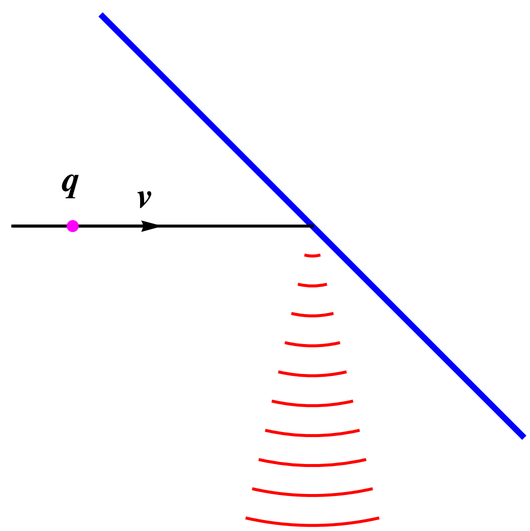

Figure 22.1: Transition radiation with foil tilted at 45 degrees.

Lecture 23

Formation length of radiation and coherent effects

Problem 23.1. *Find the vector potential for the radiation from a magnet of length $L \ll \rho/\gamma$. Hint: introduce the bending angle θ and consider the passage through the magnet as an instantaneous change in the direction of motion of the particle (see the transition radiation derivation).*

Problem 23.2. *Calculate the integral*

$$F(\omega) = \int d^3r' d^3r'' \lambda(\boldsymbol{r}')\lambda(\boldsymbol{r}'') \cos\left(\omega \frac{\boldsymbol{n} \cdot (\boldsymbol{r}' - \boldsymbol{r}'')}{c}\right) \tag{23.1}$$

for a "pancake" distribution

$$\lambda(\boldsymbol{r}) = \delta(z) \frac{1}{2\pi\sigma_r^2} e^{-(x^2+y^2)/2\sigma_r^2} . \tag{23.2}$$

The vector $\boldsymbol{n}$ is directed at angle ψ to the z axis.

Lecture 24

Synchrotron radiation reaction force

Problem 24.1. *Find the difference $\phi - cA_s$ behind the particle ($\psi < 0$) and show that $E_s \approx 0$ in that region. [A more accurate calculation shows that actually $4\pi\epsilon_0 E_s \approx q/8\rho^2$ in that region.]*

Problem 24.2. *Find the value of $E_s(s)$ using the known intensity of the radiation given by*

$$P_0 = \frac{2r_0 mc^2\gamma^4 c}{3\rho^2}\,. \tag{24.1}$$

Does it agree with the value shown in the Figure below? If not, explain the discrepancy.

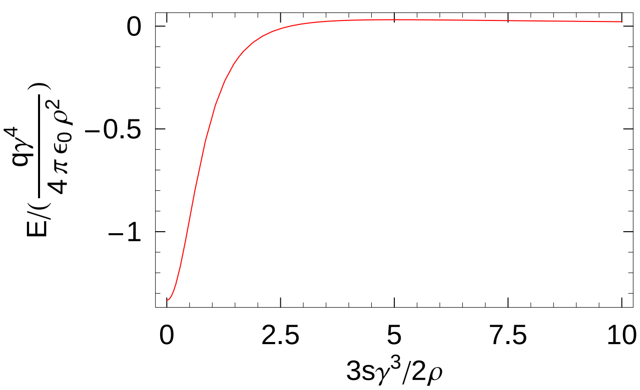

Figure 24.1: The radiation reaction field near the charge; the distance is measured in units of $3\gamma^3/2\rho$, and the field is measured in units of $q\gamma^4/4\pi\epsilon\rho^2$.

Lecture 25

Waveguides and RF cavities

Problem 25.1.*Calculate TM modes in a rectangular waveguide with cross section $a \times b$.*

Problem 25.2. *Follow up on the problem 25.1 and derive TE modes in a recutangular waveguide by applying transformation $(\boldsymbol{E}, \boldsymbol{B}) \to (c\boldsymbol{B}, -\boldsymbol{E}/c)$ to TM modes and satisfying the boundary conditions on the wall*

Problem 25.3. *Consider a point charge passing through a cylindrical cavity where a fundamental mode is excited with amplitude E_0. Calculate the maximum energy gain for the charge.*

Problem 25.4. *Estimate the electromagnetic pressure in a cavity with $E = 20$ MV/m.*

Problem 25.5. *The radius of a cylindrical cavity is changed by a small quantity δa, and the length is changed by δL. Consider this as a deformation of the cavity shape and find the frequency change using Slater's formula. Verify that the result agrees with equation*

$$\omega = j_1 \frac{c}{a}\,. \tag{25.1}$$

Problem 25.6. *Find the loss factor for the fundamental mode of the cylindrical cavity.*

Lecture 26

Laser acceleration in vacuum. Inverse FEL acceleration

Problem 26.1. *Prove that $W = 0$ even if $v = c$.*

Problem 26.2. *Prove the statement in the previous paragraph for $\beta = 1$. The modified integral to consider then becomes*

$$\Delta W_\infty = \mathrm{Re} i E_0 x_0 q e^{ikz_0} \int_{-\infty}^{\infty} \frac{d\xi}{(1+i\xi)^2} e^{-x_0^2/\omega_0^2(1+i\xi)} \, . \tag{26.1}$$

Problem 26.3. *Calculate the contribution to ΔW of the reflected part of the laser field.*

Problem 26.4. *Assume that you are given a laser with a given energy E_L, frequency ω and duration τ of the laser pulse. Optimize parameters of a laser acceleration experiment to achieve the maximum energy gain for relativistic particles. Express the energy gain in terms of E_L, ω and τ.*

Problem 26.5. *Take the the following parameters of the IFEL experiment from Ref. PRL* **92** *194801 (2005): beam energy 30 MeV, laser pulse length 2 ps, laser energy 0.5 mJ, laser focused spot size 110 μm, undulator period 1.8 cm, number of periods 3, $K = 0.6$, and estimate the amplitude of the energy modulation of the beam.*